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CONTROL OF THE MOTION OF A CYLINDRICAL BODY IN A VISCOUS MEDIUM FOR OPTIMAL ENERGY CONSUMPTION†

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Displacements of a cylindrical body in a viscous medium are considered, with a view to determining the optimal displacements, in the sense of minimum energy consumption, for a given time and distance. An Euler–Lagrange variational procedure is used to find the necessary optimum conditions, formulated as differential equations for local phases of the optimal motion of the cylinder. Using these equations it can be shown that the problem has two extremal solutions. The first corresponds to motion of the cylinder at constant velocity, preserving its vertical orientation. The second involves an intermediate stage during which the cylinder is moving in a horizontal position. Before and after that stage the velocity of the cylinder's centre of mass undergoes jumps, the number of jumps depending on the relative elongation of the cylinder. At the initial and final times the cylinder has unbounded angular velocity, but its centre of mass is moving at a finite velocity. A computational experiment has shown that if the distance of the displacement does not exceed a certain critical value, the first extremal solution is optimal. Otherwise the second extremal solution is optimal. This critical value depends only on the geometry of the cylinder.

1. FORMULATION OF THE PROBLEM

Consider the displacements of a homogeneous cylindrical body in a viscous medium, with the upper point of the body's axis (henceforth referred to briefly as the engagement point) sliding along the horizontal. The time and distance of the displacement are given; the cylinder is oriented vertically at the beginning and end of the motion. Our problem is to determine how the cylinder should move so as to minimize the energy needed to overcome drag. Similar problems have been considered in studies of dynamic optimization as applied to investigating the energy aspects of underwater bipedal locomotion [1].

The mathematical formulation is as follows. A cylinder moving in a viscous medium is subject to drag [2], which is a force applied at the centre of mass parallel to the direction of the latter's velocity of motion. It includes components due to friction and pressure. The total drag [2, 3] is calculated from the formula

$$\mathbf{D} = -C_D \rho S V \mathbf{V}/2 \tag{1.1}$$

where ρ is the density of the liquid, $\mathbf{V}$ is the velocity of the cylinder's centre of mass, V is the magnitude of the velocity, S is the area of the projection of the cylinder onto a plane perpendicular to the vector $\mathbf{V}$ and C_D is the drag coefficient.

The area S is uniquely defined by the cylinder's angle of attack α, i.e. the angle between a vector directed from the centre of mass of the cylinder to the engagement point and the vector $\mathbf{V}$ (the counterclockwise direction is taken as positive).

We will now introduce several restrictions which endow the drag with a structure which renders the relevant extremum problem amenable to analytical investigation. First, the viscous medium is assumed to be incompressible. Second, the cylinder moves in a volume of liquid which is either very extended or is enclosed within rigid boundaries. Under these conditions C_D is a function of the angle of attack and the Reynolds number only [3]. Optimum laws of motion for the cylinder will be sought for a range of Reynolds numbers in which, for a fixed incidence, the drag coefficient remains practically constant. One such range is a fairly long left half-neighbourhood of the number $\mathrm{Re} = 2 \times 10^5$, provided that the cylinder is of sufficiently large relative elongation [3]. In the situation described C_D is a function of the angle of attack only.

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Under the above assumptions, the drag (1.1) is characterized, apart from a constant factor, by the quantity $-C_D(\alpha)S(\alpha)V\mathbf{V}$, which we will now proceed to calculate. The position of the cylinder is described by generalized coordinates ξ and φ (Fig. 1). The first coordinate defines the linear displacement of the engagement point, and the second angular position of the cylinder. The coordinates of the centre of mass of the cylinder are then defined by the formulae

$$x = \xi - l\sin\varphi, \quad y = l\cos\varphi \tag{1.2}$$

where l is the half-length of the cylinder. Differentiating (1.2) with respect to time, we obtain the coordinates of the velocity vector of the centre of mass

$$\dot{x} = \upsilon - l\omega\cos\varphi, \quad \dot{y} = -l\omega\sin\varphi \tag{1.3}$$

where υ is the linear velocity of the engagement point and ω is the angular velocity of the cylinder.

The area S_b of the projection of the lateral face of the cylinder onto a plane perpendicular to the velocity vector of the centre of mass may be calculated as the absolute value of the scalar product of the vectors $b\mathbf{E}$ and $V^{-1}\mathbf{V}^{\perp}$, where $\mathbf{E}$ and $\mathbf{V}^{\perp}$ are the vectors

$$\mathbf{E} = (-\sin\varphi, \cos\varphi), \quad \mathbf{V}^{\perp} = (-\dot{y}, \dot{x}) \tag{1.4}$$

and $b = 2ld$ is a constant, d being the diameter of the cylinder. Using (1.3) and (1.4), we obtain the formula

$$S_b = bV^{-1}|p|, \quad (p = \upsilon\cos\varphi - l\omega) \tag{1.5}$$

Similarly, starting with the vectors $a\mathbf{E}$ and $V^{-1}\mathbf{V}^{\perp}$, where, by (1.4), $\mathbf{E}^{\perp} = (\cos\varphi, \sin\varphi)$ and $a = d^2/4$, one can calculate the area of the projection of the ends of the cylinder on the same plane

$$S_a = aV^{-1}|q|, \quad (q = \upsilon\sin\varphi) \tag{1.6}$$

Adding (1.5) and (1.6), we obtain an expression for the required area $S(\alpha)$, and the drag is given by

$$\mathbf{D} = -C_D\rho(b|p| + a|q|)\mathbf{V}/2 \tag{1.7}$$

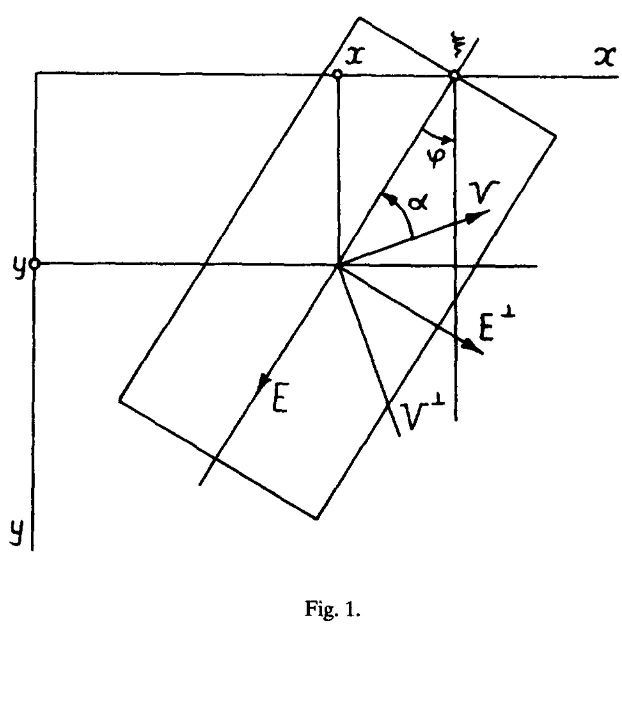

Fig. 1.

The work performed to overcome the drag over a time interval dt is equal to the scalar product of the vectors (1.7) and $\mathbf{V}dt$. Hence we obtain the following expression for the power

$$\dot{W} = C_D \rho V^2 (b|p| + a|q|)/2 \tag{1.8}$$

where, by (1.3)

$$V^2 = \upsilon^2 - 2\upsilon l\omega\cos\varphi + (l\omega)^2 \tag{1.9}$$

As a result we obtain the following dynamic optimization problem.

Problem. Find functions $\upsilon(t)$ and $\omega(t)$ that minimize the terminal function $A(t_k)$ subject to the following dynamic constraints

$$\dot{A} = C_D(\alpha)V^2(b|p| + a|q|), \quad A(0) = 0$$
$$\dot{\xi} = \upsilon, \quad \xi(0) = 0, \quad \dot{\varphi} = \omega, \quad \varphi(0) = 0 \tag{1.10}$$

and boundary conditions

$$\xi(t_k) = \xi_k, \quad \varphi(t_k) = 0 \tag{1.11}$$

The angle of attack is calculated from the formula

$$\alpha = (\pi/2)\operatorname{sign}\varphi - \operatorname{arctg}\left(\frac{l\omega\sin\varphi}{\upsilon - l\omega\cos\varphi}\right) \tag{1.12}$$

Thus, we have to find programmes $\upsilon(\cdot)$ and $\omega(\cdot)$ to vary the linear velocity of the cylinder's engagement point and angular velocity which, first, will solve the boundary-value problem (1.11) and, second, will minimize the energy consumption $W(t_k)$.

2. DERIVATION OF THE EQUATIONS FOR THE LOCAL PHASES OF OPTIMAL MOTIONS OF THE CYLINDER

In this section we will first investigate the necessary conditions for an optimum in the above dynamic optimization problem. We will then derive differential equations for the local phases of the optimal motions of the cylinder. These equations will ultimately enable us to organize computations for determining the optimal values of the generalized velocities υ and ω at any given time.

The problem formulated in Section 1 is solved by using an Euler–Lagrange variational procedure [3], namely, we set up the Hamiltonian

$$H = \lambda_0 C_D V^2 (b|p| + a|q|) + \lambda_1\upsilon + \lambda_2\omega \tag{2.1}$$

and the conjugate system

$$-\dot{\lambda}_0 = \partial H/\partial A = 0, \quad \lambda_0(t_k) = \partial\Phi/\partial A(t_k)$$
$$-\dot{\lambda}_1 = \partial H/\partial\xi = 0, \quad \lambda_1(t_k) = \partial\Phi/\partial\xi(t_k) \tag{2.2}$$
$$-\dot{\lambda}_2 = \partial H/\partial\varphi, \quad \lambda_2(t_k) = \partial\Phi/\partial\varphi(t_k)$$

The functional in these equations is

$$\Phi = A(t_k) + \nu_1(\xi(t_k) - \xi_k) + \nu_2\varphi(t_k) \tag{2.3}$$

where ν_1 and ν_2 are constants chosen so as to ensure that the boundary conditions (1.11) are satisfied.

Analysis of the conjugate system (2.2) and the functional (2.3) shows that $\lambda_0 = 1$, $\lambda_1 = \text{const}$. Hence the Euler–Lagrange equations are

$$\frac{\partial H}{\partial \upsilon}=\frac{\partial \dot{A}}{\partial \upsilon}+\lambda_1=0,\quad \frac{\partial H}{\partial \omega}=\frac{\partial \dot{A}}{\partial \omega}+\lambda_2=0 \tag{2.4}$$

where $\dot{A}$, as before, is the right-hand side of Eq. (1.10).

System (2.4) enables us to determine the Lagrange multipliers λ_1 and λ_2. Substituting the expression for λ_1 into the second equation of the conjugate system (2.2) we obtain the integral

$$\partial \dot{A}/\partial \upsilon=\text{const}=C_1 \tag{2.5}$$

Since the Hamiltonian maintains a constant value along optimal trajectories, there is a second integral: $H = C_3$. Developing this equality in accordance with (2.4), we express the second integral as follows:

$$H=\dot{A}-\upsilon\partial \dot{A}/\partial \upsilon-\omega\partial \dot{A}/\partial \omega=C_3 \tag{2.6}$$

Euler's theorem for homogeneous functions implies the formula

$$\upsilon\partial \dot{A}/\partial \upsilon+\omega\partial \dot{A}/\partial \omega=3\dot{A}$$

from which it follows, by (2.6), that $H = -2\dot{A} = C_3$. Thus, the power in the optimal displacements is a constant: $\dot{A} = C_2 = -C_3/2$.

Combining this fact with Eq. (2.5), we can establish the truth of the following.

Theorem. In optimal displacements of the cylinder, the power and its derivative with respect to the velocity of the engagement point keep constant values

$$\partial \dot{A}/\partial \upsilon=C_1,\quad \dot{A}=C_2 \tag{2.7}$$

Based on formulae (1.5), (1.6), (1.9), (1.10) and (1.12), this system determines the optimal generalized velocities as implicit functions of the cylinder's angular position: $\upsilon = f(\varphi, C_1, C_2)$, $\omega = g(\varphi, C_1, C_2)$. Thus, the optimal evolution of the cylinder must be governed by the system of equations

$$\dot{\xi}=f(\varphi,C_1,C_2),\quad \dot{\varphi}=g(\varphi,C_1,C_2) \tag{2.8}$$

The construction of an optimal phase trajectory $(\xi(t), \upsilon(t), \varphi(t), \omega(t))$ of the cylinder involves difficulties, partly computational in nature. One difficulty is obvious: the constants C_1 and C_2 are unknown, and their determination requires further investigation. A second difficulty is as follows: there is practically no hope of devising a regular procedure to solve system (2.7) for the generalized velocities, lacking any prior information as to the domains in which they may vary.

Thus, it is of the utmost importance to replace the generalized velocities of the cylinder by other characteristics, so as to overcome the second of these difficulties.

We shall describe one possible approach to this task. Equations (2.7) will be reduced to a form in which computations to determine the current optimal values of the velocities υ and ω are more conveniently organized. This will be done by introducing a new angle γ, defined by the conditions

$$p=V\cos\gamma,\quad q=V\sin\gamma \tag{2.9}$$

Let us express the integrals (2.7) for system (2.8) of optimal motions of the cylinder in terms of V and γ. To do this we will need the angle φ_m defined by

$$b=N\cos\varphi_m,\quad a=N\sin\varphi_m \tag{2.10}$$

The expression for $\dot{A}$ now becomes

$$\dot{A}=s_pC_DNV^3\cos(\gamma-s\varphi_m)=C_2 \tag{2.11}$$

where $s_p = \text{sign}\,p$, $s_q = \text{sign}\,q$, $s = s_ps_q$.

Differentiating the identity $V^2 = p^2 + q^2$ with respect to υ and using (1.5), (1.6) and the definition of γ in (2.9), we obtain a relationship from which the derivative may be expressed as

$$\partial V/\partial \upsilon = \cos(\gamma - \varphi) \tag{2.12}$$

Similarly, differentiating the identity tg $\gamma = q/p$ with respect to υ, we obtain

$$\partial\gamma/\partial\varphi = V^{-1}\sin(\varphi - \gamma) \tag{2.13}$$

Differentiation of $\dot{A}$, as defined by (2.11), with respect to υ, using formulae (2.12) and (2.13), now yields an expression for the other integral of system (2.8) of optimal motions of the cylinder

$$\partial A/\partial\upsilon = s_p C_D N V^2 (2\cos(\varphi - s\varphi_m) + \cos(2\gamma - \varphi - s\varphi_m)) +$$
$$+ C_2 V^{-2} l\omega \sin\varphi (d\ln C_D / d\alpha) = C_1 \tag{2.14}$$

Thus, the integrals (2.7) of system (2.8) have been transformed to the form (2.14) and (2.11).

We introduce the following function

$$T = T(V) = C_2(NC_D)^{-1}V^{-3} \tag{2.15}$$

In accordance with definitions (2.9) and (2.10), the integral (2.11) may be written in the form

$$\cos(\gamma - s\varphi_m) = s_p T(V) \tag{2.16}$$

Formula (2.16), considered as an equation in the angle γ, yields the following formula

$$\gamma = s\varphi_m + \sigma\arccos(s_p T) \tag{2.17}$$

where σ is a parameter that takes the values ± 1. This formula and the equality

$$2\gamma - \varphi - s\varphi_m = 2(\gamma - s\varphi_m) - (\varphi - s\varphi_m)$$

imply the relation

$$\cos(2\gamma - \varphi - s\varphi_m) = (2T^2 - 1)\cos(\varphi - s\varphi_m) + 2\sigma s_p T(1 - T^2)^{1/2}\sin(\varphi - s\varphi_m)$$

which yields the following expression for the second integral (2.14)

$$g(T,\varphi) = -s_p\left(\frac{N}{C_2 C_D^2}\right)^{1/3} l\omega\sin\varphi\left(\frac{dC_D}{d\alpha}\right)T^{4/3} \tag{2.18}$$

where

$$g(T,\varphi) = (2T^2 + 1)\cos(\varphi - s\varphi_m) + 2\sigma s_p T(1 - T^2)^{1/2}\sin(\varphi - s\varphi_m) - 3RT^{2/3} \tag{2.19}$$

$$R = s_p C_1 C_2^{-2/3}(C_D N)^{-1/3}/3$$

This relationship, considered as an equation in the parameter T, will henceforth be referred to as the Fundamental Equation.

Thus, Eqs (2.8) for the optimal motion of the cylinder have been reduced to the form

$$\dot{\xi} = V\frac{\sin\gamma}{\sin\varphi}, \quad \dot{\varphi} = l^{-1}V\frac{\sin(\gamma - \varphi)}{\sin\varphi} \tag{2.20}$$

where, by (2.15), the magnitude of the velocity of the cylinder's centre of mass is given by the formula

$$V = C_2^{1/3} C_D^{-1/3} N^{-1/3} T^{-1/3} \tag{2.21}$$

The angle γ is determined from (2.17) and T is a root of the Fundamental Equation (2.18).

To integrate this system, using, say, Euler's method, one proceeds as follows. Given an angle $\varphi(t_k)$ and velocities $\upsilon(t_{k-1})$, $\omega(t_{k-1})$, one determines the root T of the Fundamental Equation. Then, using formula (2.17), one computes γ and, using formula (2.21), the linear velocity V of the centre of mass. Using this information, one can now determine the right-hand sides of Eqs (2.20), which give the optimal displacements of the cylinder. Finally, using the formulae for Euler's method, one finds the generalized coordinates of the cylinder, $\xi(t_{k+1})$, $\varphi(t_{k+1})$.

It has been rigorously established that the root of the Fundamental Equation (2.18) should be sought in the interval $0 \leqslant T \leqslant 1$. Thus the entire problem reduces to determining the constants C_1, C_2 and σ. This will be discussed in Section 3.

3. CONSTRUCTION OF EXTREMAL PHASE TRAJECTORIES

The first extremal programme for the displacement of a cylinder in a viscous medium. Analysis of the necessary optimum conditions (2.7), taking (1.10), (1.5), (1.6), (1.9) and (1.12) into account, reveals a simple extremal trajectory, corresponding to displacement of the cylinder with its vertical orientation maintained, at a constant velocity $\upsilon = \xi_k/t_k$. In that case

$$C_1 = 3C_D(\pi/2)b(\xi_k/t_k)^2$$
$$C_2 = C_D(\pi/2)b(\xi_k/t_k)^3$$

The second extremal programme. A more detailed investigation of the necessary optimum conditions leads to the conclusion that there is another, qualitatively different, extremal, provided the given range of the displacement is large enough. The corresponding programme of cylinder displacement includes a stage in which it moves in a horizontal position.

Numerical experiment has shown that the structure of the second extremal is the same as that of an extremal corresponding to the following value of the drag coefficient

$$C_D(\alpha) = (C_D(0) + C_D(\pi/2))/2 = C_D^0 \tag{3.1}$$

provided that the cylinder is of sufficiently high relative elongation.

This is the case, for example, for cylindrical cans with fuel rods and the decay tank in certain nuclear power stations. The relative elongation of such cans is $2l/d = 2.4$. In the specific range of Reynolds numbers indicated in Section 1, the drag coefficient for the cylinder is $C_D(\pi/2) \simeq 0.69$ (for motion with the vertical orientation maintained) and $C_D(0) \simeq 0.84$ (for motion in a horizontal position [3]). Consequently, the coefficient $C_D(\alpha)$ may differ from the constant value $C_D^0 = 0.76$ by at most 11%. These conditions ensure a structural analogy between the extremals as described. The advantage of the extremal corresponding to (3.1) is that it can be described by analytical means. That is why it will receive preference in what follows, although the results of the actual computations relate to the original situation.

We thus consider the following auxiliary boundary-value problem. Consider system (2.20) when $C_D(\alpha) = C_D^0$ as in (3.1). The velocity V of the centre of mass and the angle γ are determined from (2.17) and (2.21), the constant C_2 being replaced by $C_2(C_D^0)^{-1}$. The parameter T ($0 \leqslant T \leqslant 1$) is a root of the Fundamental Equation $g(T, \varphi) = 0$ with the constant R defined by (2.19) and the constants C_i replaced by $C_i(C_D^0)^{-1}$, $i = 1, 2$.

Note that the parameter σ takes one of the values ± 1.

We have to choose σ, C_1 and C_2 so that the solution of the Cauchy problem $\xi(0) = 0$, $\varphi(0) = 0$ for system (2.20) will satisfy the boundary conditions (1.11).

Let us return to the two integrals (2.7) determined in Section 2, which in the present case have the form

$$V^2(b|p| + a|q|) = C_2$$
$$2(pp_\upsilon + qq_\upsilon)(b|p| + a|q|) + V(s_p b p_\upsilon + s_q a q_\upsilon) = C_1 \tag{3.2}$$

At the initial instant, $\varphi = 0$. Thus, by (1.9), (1.5) and (1.6), the linear velocity of the centre of mass of the cylinder is such that

$$V^2 = p^2 + q^2 = \upsilon^2 - 2l\omega\upsilon + (l\omega)^2 = p^2$$

whence it follows that $q = 0$. The integrals (3.2), with (1.5) and (1.6), may now be written in the form $bV^3 = C_2$, $3bV^2 = s_pC_1$. Hence the constants C_1 and C_2 turn out to be related by the formula

$$C_1 = 3s_p b^{1/3} C_2^{2/3} \tag{3.3}$$

Hence, in view of the notation (2.19), $R = (\cos\varphi_m)^{1/3}$. Consequently, the Fundamental Equation has a root $T = \cos\varphi_m$. This is possible only if

$$\sigma s_q = -1 \tag{3.4}$$

Analysis of the first and second derivatives of the function $T(\varphi)$, defined implicitly by the Fundamental Equation (2.18), with respect to the generalized coordinate φ, yields

$$s = s_p s_q = 1 \tag{3.5}$$

Thus, by (3.4) and (3.5), confining our attention to small angular displacements, we can write the Fundamental Equation as

$$g(T,\varphi) = (2T^2+1)\cos(\varphi-\varphi_m) - 2T(1-T^2)^{1/2}\sin(\varphi-\varphi_m) - 3\cos^{1/3}\varphi_m T^{2/3} = 0 \tag{3.6}$$

The angle γ is calculated from the formula $\gamma = \varphi_m - s_p \arccos(s_p(T)$.

Since $\upsilon(0) > 0$, it follows that $s_q = 1$, so that, by (3.5), $s_p = 1$. Thus the formula for γ becomes

$$\gamma = \varphi_m - \arccos T \tag{3.7}$$

Applying l'Hôpital's rule to the right-hand sides of Eqs (2.20), using (2.21), (3.6) and (3.7), we conclude that the velocities υ and ω are unbounded in the neighbourhood of the starting time. Nevertheless, the velocity of the centre of mass remains bounded.

As the time increases, the angle φ also increases, reaching the value $\varphi_p = \varphi_m - \arccos(\cos^{1/3}\varphi_m)$ at a certain time. When that occurs we have $T(\varphi_p) = 1$, and, for the Fundamental Equation to be solvable at later times as well, we must set $\sigma = 1$. The parameter $T(\varphi)$ then begins to decrease, reaching a certain minimum value; it then increases and, when the angle $2\varphi_m$ is reached, becomes equal to $\cos\varphi_m$. At that point the angular velocity of the cylinder vanishes.

We shall need the following iterative process, which involves a sequence of pairs of numbers $\{R_k, \varphi_{pk}\}$, $k = 2, 3, \ldots$. Let $R_2 = -\sin\varphi_m$, $\varphi_{p2} = \pi/2$ and suppose that the pairs with numbers $3, \ldots, k$ have been constructed. We then define $R_k+1 = -\chi(T_k)$, where T_k is the greatest root of the equation

$$\frac{2T\chi_k(T)(2T^2-1)(1-\chi_k^2(T))^{1/2}}{(1-T^2)^{1/2}} + \frac{R_k}{T^{1/3}} = 0$$

in the interval [0, 1]. We have used the notation

$$\chi_k(T) = -T^{2/3}(8T^2+1)^{-1}(3R_k(2T^2+1) + 2T^{1/3}(1-T^2)^{1/2}(8T^2-9R_k^2T^{4/3}+1)^{1/2})$$

The angle $\varphi_{p,k+1} = \pi - \varphi_m - \arccos\chi_k(T_k)$.

It can be verified that such $g(T_k, \varphi_{pk}) = \partial g/\partial T(T_k, \varphi_{pk}) = 0$, $k = 3, \ldots$.

Let n be a number such that $\varphi_{p,n+1} \leqslant 2\varphi_m < \varphi_{pn}$. As the cylinder passes through the angle $2\varphi_m$, the sign of the parameter s_p changes, and the constant R also changes. Now $s_p = -1$, $s_q = 1$, $R = R_n$. This value of R is maintained until the cylinder turns through the angle φ_{pn}. The parameter T increases to 1 as the angle φ increases. In the angular sector $(\varphi_{pn}, \varphi_{p,n-1}]$ we have $R = R_{n-1}$, T increases to 1 as the angle φ increases, and so on. Finally, when the cylinder passes through the angle φ_{p3} the constant R takes the value $-\sin\varphi_m$ and at that point takes a horizontal position.

After the cylinder has become horizontal, it continues to move steadily at a constant velocity $\upsilon_s = (C_2/b)^{1/3}$. This follows from the first formula of (3.2). The constant C_2 is determined from the first condition of (1.11), written in the form $\upsilon_s t_s = \xi_k - 2\xi(\Delta)$, where Δ is the time needed for the cylinder to turn from vertical to horizontal, $t_s = t_k - 2\Delta$.

With the elapse of time t_s, the cylinder begins to resume its original, vertical position. This takes place in accordance with the programme described above, but reversing the signs of the parameters σ, s_p and s_q compared with the first manoeuvre.

The problem has no extremal solutions other than those described. It remains to ascertain under what conditions one or the other construction is optimal. It is clear that the possible energy gain when the cylinder is moving in the second extremal programme is due to the intermediate, steady-state phase of the motion. In this phase the drag of a cylinder of sufficiently high relative elongation reaches its minimum, so that, for the same power, the cylinder may move at a higher velocity.

4. COMPUTATIONAL EXPERIMENT

To answer the question posed above, a program was written to simulate the motion of the cylinder along an optimal phase trajectory $(\xi^0(t), \varphi^0(t), \upsilon^0(t), \omega^0(t))$. The results are exhibited below for initial data $d = 151$ cm and $l = 182.5$ cm. The linear velocity of the cylinder in the first extremal program was taken as 10 cm/s.

Let Q be the ratio of the amounts of energy expended in the second and first extremal programs.

It has been shown that over runs of at most 2900 cm the first extremal construction requires less energy. In longer runs, however, the second program becomes more suitable. The data are presented in the following table

ξ_k	2900	3240	4340	7850
Q	1	0.9	0.7	0.5

If the run is increased without limit, the coefficient Q approaches the value $aC_D(0)/(bC_D(\pi/2))$, which is proportional to the ratio of the area of an end of the cylinder to the area of the cross-section through its vertical axis.

The computational experiment has shown that the ranges just indicated are independent of the velocity of the cylinder in the first extremal program; they depend solely on its geometry.

In Figs 2–5, various characteristics of the cylinder are plotted against its angular position φ: the angular velocity ω and linear velocity υ of the engagement point (Fig. 2); the velocity V of the centre of mass and the angle of attack α (Fig. 3); the components $\dot{x}, \dot{y}$ of the velocity of the centre of mass (Fig. 4); and, finally, the parameter T and the area of the projection of the cylinder on a plane perpendicular to the velocity vector of the centre of mass (Fig. 5); all these plots refer to the phase of optimal motion of the cylinder from vertical to horizontal.

The second extremal program stipulates a stage during which the cylinder is moving in a horizontal position. Before and after this stage, the velocity of the centre of mass of the cylinder experiences a jump three times. This takes place at angles $2\varphi_m$, φ_{p3}, $\pi/2$. The behaviour of the generalized velocities of the cylinder and its angle of attack is similar. At the initial and final times the angular velocity of the cylinder, as well as the linear velocity of the engagement point, are unbounded, but the velocity of the centre of mass remains finite.

It is clear that in order to implement the second extremal program one must apply fairly irregular external forces and torques to the cylinder. Let these be, say, a horizontal force applied at the engagement point and a torque about the same point. Then both force and torque are unbounded at the beginning and end of the process. Three times before the part of the motion in the horizontal position and three times after it, the controlling torque acts impulsively on the cylinder, at the times

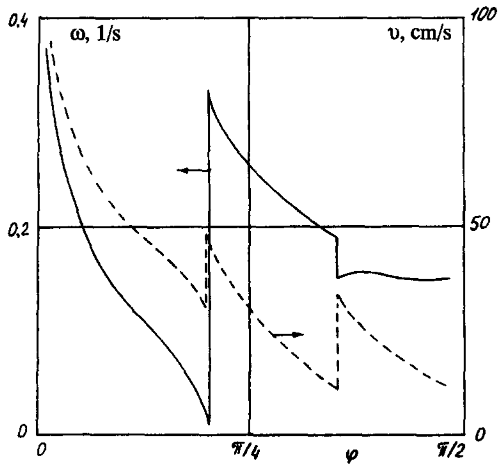

Fig. 2.

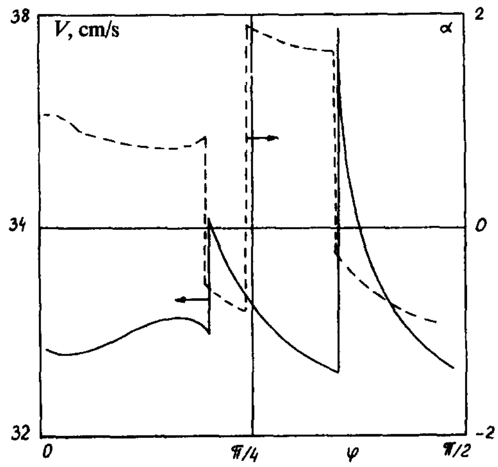

Fig. 3.

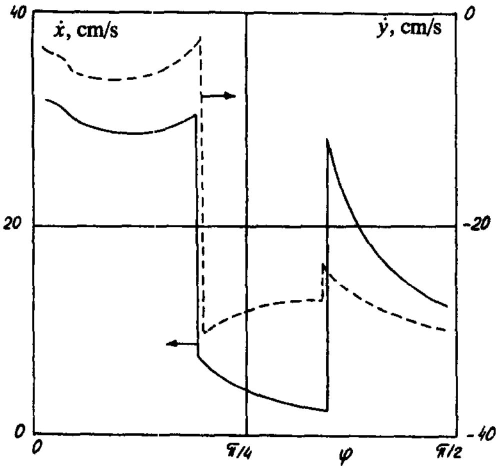

Fig. 4.

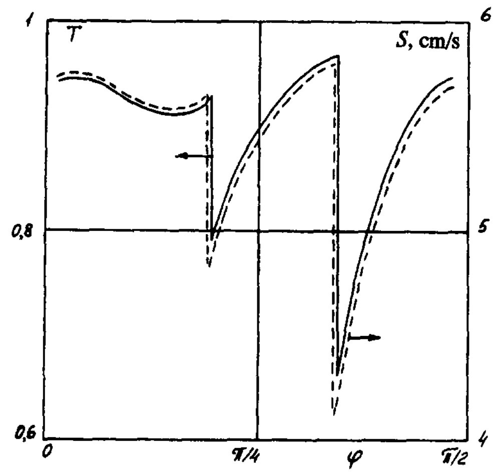

Fig. 5.

$$t_1, t_2, \Delta, \Delta+t_s, \Delta+t_s+t_1, \Delta+t_s+t_2 \qquad (4.1)$$

where the times t_1 and t_2 are determined from the conditions $\varphi(t_1) = 2\varphi_m$, $\varphi(t_2) = \varphi_{p3}$. The behaviour of the force applied at the engagement point is similar.

This behaviour of the controls on the cylinder determines our problem's category in the field of dynamic optimization: it is an irregular problem [4, 5], solvable by one of the techniques developed in [6, 7]. In addition, the problem is interesting for the fact that it provides a meaningful model in which the so-called singular manifold (2.20) has a discontinuous structure, explaining the presence of pulsed components of the controlling force and torque at the times (4.1). It is these pulsed effects that enable the phase portrait of the cylinder to remain on the singular manifold across the latter's discontinuities.

Apart from the experiment described, another experiment has been performed in order to determine the range of possible relative elongations of the cylinder in which the second extremal construction becomes optimal from some range ξ_k on. As shown, this is the case if the diameter of the cylinder does not exceed a certain quantity of order $6.08l/\pi$, where $2l$ is its length. The range ξ_k increases without limit as the diameter approaches this quantity.

Finally, we present some data on the number of jumps in the velocity of the centre of mass of the cylinder when confined to the horizontal position, depending on the relative elongation $2l/d$. The numerical experiment was carried out for a cylinder of length $2l = 365$ cm. The existence has been established of constants $d_2 \simeq 239.5$ cm, $d_3 \simeq 124$ cm, $d_4 \simeq 100$ cm, ..., such that the number of these jumps is two if $d_2 < d$, three if $d_3 < d < d_2$, four if $d_4 < d < d_3$, etc.

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